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Inhomogeneous Six-Vertex Model with Domain Wall Boundary Conditions and Bethe Ansatz

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In this note, we consider the six-vertex model with domain wall boundary conditions, defined on a $M \times M$ lattice, in the inhomogeneous case where the partition function depends on $2M$ inhomogeneities λ_j and μ_k. For a particular choice of the set of λ_j we find a new determinant representation for the partition function, which allows evaluation of the bulk free energy in the thermodynamic limit. This provides a new connection between two types of determinant formulae. We also show in a special case that spin correlations on the horizontal line going through the center coincide with the ones for periodic boundary conditions.

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1. Introduction

The six-vertex model was first introduced in [1]. It was solved exactly by E. Lieb [2] and B. Sutherland [3] in 1967 by means of Bethe Ansatz for periodic boundary conditions (PBC). Later the six-vertex model was studied for different boundary conditions [4],[5],[6]. Domain wall boundary conditions were introduced in 1982 [7]. These boundary conditions are interesting because they allow to derive determinant representations for correlation functions [8] and the same boundary conditions help to enumerate alternating sign matrices [9], [10]. Recently the bulk free energy was calculated for these boundary conditions [11].

In this paper we show that for special choices of inhomogeneities, one can compute the free energy and some correlation functions of the system. This observation might be useful because we expect some properties of the model to be independent of the inhomogeneities i.e. to depend only on the anisotropy parameter. In the simplest situation, the correlation functions coincide with the ones for periodic boundary conditions.

2. Inhomogeneities and Bethe Ansatz

In this section we define the inhomogeneous six-vertex model with domain wall boundary conditions and choose the spectral parameters (inhomogeneities) to satisfy Bethe Ansatz equations; this will imply special properties of the partition function.

We now introduce the notations. Given a $M \times M$ square lattice with spectral parameters λ_i and μ_k attached to the lines and columns, one defines the usual Boltzmann weights a, b, c to be

$$\begin{aligned} a(\lambda,\mu) &= \sinh(\gamma(\lambda-\mu+i/2)) \\ b(\lambda,\mu) &= \sinh(\gamma(\lambda-\mu-i/2)) \\ c(\lambda,\mu) &= \sinh(i\gamma) \end{aligned} \tag{2.1}$$

where γ is the anisotropy. We have *fixed* boundary conditions for the external edges: horizontal (resp. vertical) external edges are outgoing (resp. incoming). The partition function is denoted by $Z_M(\{\lambda_j\},\{\mu_k\})$. Using recursion relations satisfied by the Z_M, one can show that the following determinant formula holds [12], [13]:

$$\begin{aligned} Z_M(\{\lambda_j\},\{\mu_k\}) =& \frac{\prod_{1\le j,k\le M}\sinh(\gamma(\lambda_j-\mu_k+i/2))\sinh(\gamma(\lambda_j-\mu_k-i/2))}{\prod_{1\le j<j'\le M}\sinh(\gamma(\lambda_j-\lambda_{j'}))\prod_{1\le k<k'\le M}\sinh(\gamma(\mu_k-\mu_{k'}))} \\ & \det_{1\le j,k\le M}\left[\frac{\sinh(i\gamma)}{\sinh(\gamma(\lambda_j-\mu_k+i/2))\sinh(\gamma(\lambda_j-\mu_k-i/2))}\right] \end{aligned} \tag{2.2}$$

In the homogeneous case this represention was used in order to evaluate the bulk free energy in the thermodynamic limit [11]. In a special inhomogeneous case we shall use another determinant represention to evaluate the bulk free energy in the thermodynamic limit. Let us define our special case. We choose the spectral parameters in relation to the Bethe Ansatz. In order to do that it is convenient to introduce the formalism of the Algebraic Bethe Ansatz. The Boltmann weights are encoded into the *L-matrix*

$$L(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix} \tag{2.3}$$

We next introduce the *monodromy matrix* $T(\lambda)$ which is an operator acting on $\mathbb{C}^{2^M} \otimes \mathbb{C}^2$ (physical space times auxiliary space) defined by

$$T(\lambda; \mu_1, \ldots, \mu_M) = L(\lambda - \mu_M) L(\lambda - \mu_{M-1}) \ldots L(\lambda - \mu_1) \tag{2.4}$$

where $L(\lambda - \mu_k)$ acts on the k^{th} factor of the tensor product in the physical space, and the auxiliary space. As an operator on the two-dimensional auxiliary space, $T(\lambda)$ can be written as

$$T(\lambda) = \begin{pmatrix} \mathbf{A}(\lambda) & \mathbf{B}(\lambda) \\ \mathbf{C}(\lambda) & \mathbf{D}(\lambda) \end{pmatrix} \tag{2.5}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are operators on the physical space $\mathbb{C}^{2^M}$. The usual *transfer matrix* corresponding to periodic boundary conditions is defined as

$$\mathbf{T}(\lambda) = \mathbf{A}(\lambda) + \mathbf{D}(\lambda) \tag{2.6}$$

We shall make use of this operator later. Here, our fixed boundary conditions imply the following formal expression for the partition function [7]:

$$Z_M(\{\lambda_j\}, \{\mu_k\}) = \langle \downarrow \big| \mathbf{B}(\lambda_1; \mu_1, \ldots, \mu_M) \ldots \mathbf{B}(\lambda_M; \mu_1, \ldots, \mu_M) \big| \uparrow \rangle \tag{2.7}$$

where $\big| \uparrow \rangle \equiv \binom{1}{0}^{\otimes M}$ (resp. $\big| \downarrow \rangle \equiv \binom{0}{1}^{\otimes M}$) is the state with all spins up (resp. down).

Let us specify inhomogeneities. We choose the $\{\lambda_j\}$ to be divided into two sets $\{\lambda_j^1 = \lambda_j,\, j = 1, \ldots, m^1\}$ and $\{\lambda_j^2 = \lambda_{j+m^1},\, j = 1, \ldots, m^2\}$ which each satisfy the *Bethe Ansatz Equations*:

$$\prod_{\substack{j'=1 \\ j' \neq j}}^{m^\alpha} \frac{\sinh(\gamma(\lambda_j^\alpha - \lambda_{j'}^\alpha + i))}{\sinh(\gamma(\lambda_j^\alpha - \lambda_{j'}^\alpha - i))} = \prod_{k=1}^{M} \frac{\sinh(\gamma(\lambda_j^\alpha - \mu_k + i/2))}{\sinh(\gamma(\lambda_j^\alpha - \mu_k - i/2))} \tag{2.8}$$

with $\alpha = 1, 2$. We define the left and right states

$$\begin{aligned} \big\langle 1 \big| &= \big\langle \uparrow \big| \, \mathbf{C}(\lambda_1^1) \dots \mathbf{C}(\lambda_{m^1}^1) \\ \big| 2 \big\rangle &= \mathbf{B}(\lambda_1^2) \dots \mathbf{B}(\lambda_{m^2}^2) \, \big| \uparrow \big\rangle \end{aligned} \tag{2.9}$$

and the *flip operator* $\mathbf{R}$ to be the operator on the physical space that flips all arrows: in terms of the usual Pauli matrices, $\mathbf{R} = \prod_{k=1}^{M} \sigma_k^x$. Now using the overall invariance of the monodromy matrix under flip: $[T(\lambda), \mathbf{R}\sigma^x] = 0$ (where the additional σ^x acts on the auxiliary space), one finds that $\mathbf{R}\,\mathbf{B}(\lambda)\,\mathbf{R} = \mathbf{C}(\lambda)$ and therefore we can rewrite formula (2.7) in terms of the states we have defined:

$$Z_M(\{\lambda_j\}, \{\mu_k\}) = \big\langle 1 \big| \, \mathbf{R} \, \big| 2 \big\rangle \tag{2.10}$$

We now consider the situation where M is even and $m^1 = m^2 = M/2$. As proven in the appendix (**A.3**), the Bethe state $\big| 2 \big\rangle$ is an eigenstate of $\mathbf{R}$ (with eigenvalue ± 1). At this point we use orthogonality of Bethe states (**A.2**) to conclude that

$$Z_M(\{\lambda_j\}, \{\mu_k\}) = \pm \delta_{\{\lambda_j^1\}, \{\lambda_j^2\}} \big\langle 1 \big| 2 \big\rangle \tag{2.11}$$

The non-zero scalar product (square of the norm) is given by the following formula [7], dropping the superscripts:

$$\big\langle 1 \big| 1 \big\rangle = (\sin\gamma)^{M/2} \left[\prod_{j=1}^{M/2} \mathrm{a}(\lambda_j) \mathrm{d}(\lambda_j) \right] \left[\prod_{\substack{j,j'=1 \\ j \neq j'}}^{M/2} \frac{\sinh(\gamma(\lambda_j - \lambda_{j'} + i))}{\sinh(\gamma(\lambda_j - \lambda_{j'}))} \right] \det_{1 \leq j,j' \leq M/2} \left[\frac{\partial \varphi_j}{\partial \lambda_{j'}} \right] \tag{2.12}$$

with the following definitions: $\mathrm{a}(\lambda)$ and $\mathrm{d}(\lambda)$ are the eigenvalues of $\mathbf{A}(\lambda)$ and $\mathbf{D}(\lambda)$ acting on $\big| \uparrow \big\rangle$:

$$\begin{aligned} \mathrm{a}(\lambda) &= \prod_{k=1}^{M} \sinh(\gamma(\lambda - \mu_k + i/2)) \\ \mathrm{d}(\lambda) &= \prod_{k=1}^{M} \sinh(\gamma(\lambda - \mu_k - i/2)) \end{aligned} \tag{2.13}$$

and the φ_j are the logarithms of the B.A.E. (2.8):

$$\varphi_j = i \log(\mathrm{a}(\lambda_j)/\mathrm{d}(\lambda_j)) + i \sum_{\substack{j'=1 \\ j' \neq j}}^{M/2} \log \frac{\sinh(\gamma(\lambda_j - \lambda_{j'} + i))}{\sinh(\gamma(\lambda_j - \lambda_{j'} - i))} \tag{2.14}$$

Note that the general determinant formula (2.2), in the case of 2 identical sets $\{\lambda_j^1\} = \{\lambda_j^2\}$, becomes

$$Z_M(\{\lambda_j\},\{\mu_k\}) = \frac{\prod_{\substack{1\le j\le M/2\\ 1\le k\le M}} \sinh^2(\gamma(\lambda_j-\mu_k+i/2))\sinh^2(\gamma(\lambda_j-\mu_k-i/2))}{\prod_{1\le j<j'\le M/2}\sinh^2(\gamma(\lambda_j-\lambda_{j'}))\prod_{1\le k<k'\le M}\sinh(\gamma(\mu_k-\mu_{k'}))} \det_{\substack{1\le j\le M/2\\ 1\le k\le M}}[\phi(\lambda_j-\mu_k),\psi(\lambda_j-\mu_k)] \tag{2.15}$$

with $\phi(\lambda)\equiv \sinh(i\gamma)/(\sinh(\gamma(\lambda+i/2))\sinh(\gamma(\lambda-i/2)))$ and $\psi(\lambda)=\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi(\lambda)$. Our new determinant representation (2.12) is quite different from Eq. (2.15); it is in particular much easier to study its thermodynamic limit.

3. Correlation functions

We have studied so far the function partition of the system; what about correlation functions?

It is unfortunately not possible to find such a simple expression for an arbitrary correlation function; however, if we restrict ourselves to the case of special correlations which lie on a fixed horizontal line, then one can again derive determinant formulae for them.

To be specific, let us assume again that the $\{\lambda_j\}$ consist of two identical sets $\{\lambda_j,\ j=1\dots M/2\}$. Then the probability that all arrows located at columns $k_1,\dots,k_n$ and between lines $M/2$ and $M/2+1$ are up, is given by

$$\langle \pi_{k_1}\dots\pi_{k_n}\rangle \equiv \frac{\langle\downarrow|\mathbf{B}(\lambda_1)\dots\mathbf{B}(\lambda_{M/2})\,\pi_{k_1}\dots\pi_{k_n}\,\mathbf{B}(\lambda_1)\dots\mathbf{B}(\lambda_{M/2})|\uparrow\rangle}{\langle\downarrow|\mathbf{B}(\lambda_1)\dots\mathbf{B}(\lambda_{M/2})\,\mathbf{B}(\lambda_1)\dots\mathbf{B}(\lambda_{M/2})|\uparrow\rangle} \tag{3.1}$$

where $\pi_k\equiv\frac{1}{2}(1+\sigma_k^z)=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ acts on the k^{th} space.

Similarly to what was done for the partition function, one can transform Eq. (3.1) using the flip operator $\mathbf{R}$ and find:

$$\langle\pi_{k_1}\dots\pi_{k_n}\rangle = \frac{\langle 1|\,\pi_{k_1}\dots\pi_{k_n}\,|1\rangle}{\langle 1|1\rangle} \tag{3.2}$$

In other words this is simply the usual correlation functions of the spin operators $\frac{1}{2}(1+\sigma^z)$ for the corresponding spin chain. The computation of these averages is a well-known

problem, and the general strategy to perform it is by now well understood [14]. One must use the solution of the quantum inverse scattering problem for these operators [15,16]:

$$\pi_k = \prod_{l=1}^{k-1} \mathbf{T}(\mu_l + i/2)\, \mathbf{A}(\mu_k + i/2) \prod_{l=k+1}^{M} \mathbf{T}(\mu_l + i/2) \tag{3.3}$$

and then use the fact that our state $\big|1\big\rangle$ is an eigenstate of the $\mathbf{T}(\lambda)$, as well as the commutation relations satisfied by the $\mathbf{A}(\lambda)$ and $\mathbf{B}(\lambda)$. In this way we can reduce Eq. (3.1) (resp. Eq. (3.2)) to a sum of expressions of the type $\big\langle \downarrow \big| \mathbf{B}(\tilde\lambda_1)\ldots\mathbf{B}(\tilde\lambda_{M/2})\mathbf{B}(\lambda_1)\ldots\mathbf{B}(\lambda_{M/2}) \big| \uparrow \big\rangle$ (resp. $\big\langle \uparrow \big| \mathbf{C}(\tilde\lambda_1)\ldots\mathbf{C}(\tilde\lambda_{M/2})\mathbf{B}(\lambda_1)\ldots\mathbf{B}(\lambda_{M/2}) \big| \uparrow \big\rangle$). These can finally be expressed as determinants either using Eq. (2.2), or according to the general formula for scalar products (p. 237 of [8]).

The case $n=1$ is trivial: $\big|1\big\rangle$ is an eigenstate of $\mathbf{R}$, and therefore $\langle \pi_k \rangle = 1/2$. The simplest higher correlation function is the emptiness formation probability where all k_i are nearest neighbors, in which case we obtain immediately

$$\langle \pi_{k+1}\ldots\pi_{k+n}\rangle = \prod_{l=k+1}^{k+n}\left[\mathrm{a}(\mu_l+i/2)\prod_{j=1}^{M/2}\frac{\sinh(\gamma(\mu_l-\lambda_j+i/2))}{\sinh(\gamma(\mu_l-\lambda_j-i/2))}\right]\langle 1|\prod_{l=k+1}^{k+n}\mathbf{A}(\mu_l+i/2)\,|1\rangle \tag{3.4}$$

We shall give the thermodynamic limit of this expression in a particular case.

4. Thermodynamic limit

We shall show in a particular example how to take the thermodynamic limit in formulae (2.12) and (3.4). We set all μ_k to 0 and consider the critical regime i.e. γ real. We specify the state $\big|1\big\rangle$ to be the ground state of the transfer matrices $\mathbf{T}(\lambda)$ (or of the Hamiltonian of the corresponding XXZ spin chain). In the limit $M\to\infty$ the λ_j form a continuous distribution on the real axis determined by its density $\rho(\lambda) = 1/(2\cosh(\pi\lambda))$.

Let us first consider the free energy. One can show that the undetermined sign in Eq. (2.11) is +, and therefore we have $F = -\log\big\langle 1\big|1\big\rangle$. We now analyze Eq. (2.12) in the large M limit. We see that we have

$$\begin{aligned}-F \approx M^2\Bigg[&\frac{1}{2}\int \mathrm{d}\lambda\rho(\lambda)\log[\sinh(\gamma(\lambda-i/2))\sinh(\gamma(\lambda+i/2))] \\ &+\frac{1}{4}\int \mathrm{d}\lambda\rho(\lambda)\mathrm{d}\lambda'\rho(\lambda')\log\frac{\sinh(\gamma(\lambda-\lambda'+i))}{\sinh(\gamma(\lambda-\lambda'))}\Bigg] + M\frac{1}{2}\log\sin\gamma + \log\det D\end{aligned} \tag{4.1}$$

The terms of order M^2 form the bulk free energy. In order to go further we have to analyze the behavior of the determinant. It is easy to see that the determinant of the matrix $D \equiv \left[\frac{\partial\varphi_j}{\partial\lambda_{j'}}\right]$ is dominated by its diagonal elements; the latter are, by definition of $\rho(\lambda)$,

$$\frac{\partial\varphi_j}{\partial\lambda_j} = 2\pi\frac{M}{2}\rho(\lambda_j) \tag{4.2}$$

and therefore

$$\log\det D \approx \frac{M}{2}\log(\pi M) + \frac{M}{2}\int \mathrm{d}\lambda\rho(\lambda)\log\rho(\lambda) \tag{4.3}$$

This gives us the expansion of the free energy up to linear terms in the size M. Note that similar expressions (with different densities $\rho(\lambda)$) can be found for other states, as long as they have a proper $M\to\infty$ limit.

As to the correlation functions, it is known that there is a general multiple integral representation for correlation functions in the thermodynamic limit [17] (see also [18], [19]). We shall not repeat the derivation here; let us simply mention that starting for example from (3.4), one can prove the following formula [15,20]:

$$\langle\pi_{k+1}\dots\pi_{k+n}\rangle = 2^{-n}\left(\frac{\pi}{\zeta}\right)^{n(n-1)/2}\int_{-\infty}^{+\infty}\mathrm{d}\rho_1\dots\mathrm{d}\rho_n\prod_{j<k}\frac{\sinh\pi(\rho_j-\rho_k)}{\sinh\gamma(\rho_j-\rho_k-i)} \\ \prod_{j=1}^{n}\frac{\sinh^{j-1}\gamma(\rho_j-i/2)\sinh^{m-j}\gamma(\rho_j+i/2)}{\cosh^m\pi\rho_j} \tag{4.4}$$

This result is identical to the correlation functions of the 6-vertex model with periodic boundary conditions.

Appendix A. A non-degeneracy property

We assume in this appendix that $q\equiv \mathrm{e}^{i\gamma}$ is generic (i.e. not a root of unity, except the isotropic case $q=-1$). We also assume that the spectral parameters $\{\mu_k\}$ do not form any "strings" (i.e. $\mathrm{Im}(\mu_k-\mu_l)\neq 1\quad\forall k,l$). We consider two Bethe states $|1\rangle$ and $|2\rangle$ characterized by two sets $\{\lambda_j^1,\, j=1,\dots,m^1\}$ and $\{\lambda_j^2,\, j=1,\dots,m^2\}$. Bethe states are eigenstates of the set of commuting transfer matrices $\mathbf{T}(\lambda)$, with corresponding eigenvalue

$$\mathbf{T}(\lambda)\,|\alpha\rangle = \left[\frac{\mathrm{Q}^\alpha(\lambda-i)}{\mathrm{Q}^\alpha(\lambda)}\mathrm{a}(\lambda) + \frac{\mathrm{Q}^\alpha(\lambda+i)}{\mathrm{Q}^\alpha(\lambda)}\mathrm{d}(\lambda)\right]|\alpha\rangle \tag{A.1}$$

where $\mathrm{a}(\lambda)$ and $\mathrm{d}(\lambda)$ are given by Eq. (2.13) and are independent of the state, whereas Q^α characterizes the $\{\lambda_j^\alpha\}$:

$$\mathrm{Q}^\alpha(\lambda) = \prod_{j=1}^{m^\alpha} \sinh(\gamma(\lambda - \lambda_j^\alpha)) \tag{A.2}$$

Note that the Bethe Ansatz Equations are simply the equations which ensure pole cancellation in the eigenvalue of $\mathbf{T}(\lambda)$: Res $\mathbf{T}(\lambda)\,|\alpha\rangle_{|\lambda=\lambda_j^\alpha} = 0$.

Because of the symmetry of the transfer matrices $\mathbf{T}(\lambda)$ under the flip operator $\mathbf{R}$, we are only considering states with $m^\alpha \leq M/2$.

We now assume that $|1\rangle$ and $|2\rangle$ have the same eigenvalue, that is

$$\frac{\mathrm{Q}^1(\lambda+i)}{\mathrm{Q}^1(\lambda)}\mathrm{a}(\lambda) + \frac{\mathrm{Q}^1(\lambda-i)}{\mathrm{Q}^1(\lambda)}\mathrm{d}(\lambda) = \frac{\mathrm{Q}^2(\lambda+i)}{\mathrm{Q}^2(\lambda)}\mathrm{a}(\lambda) + \frac{\mathrm{Q}^2(\lambda-i)}{\mathrm{Q}^2(\lambda)}\mathrm{d}(\lambda) \qquad \forall\lambda \tag{A.3}$$

We rewrite this as

$$\mathrm{a}(\lambda)\left[\mathrm{Q}^1(\lambda+i)\mathrm{Q}^2(\lambda) - \mathrm{Q}^2(\lambda+i)\mathrm{Q}^1(\lambda)\right] = \mathrm{d}(\lambda)\left[\mathrm{Q}^2(\lambda-i)\mathrm{Q}^1(\lambda) - \mathrm{Q}^1(\lambda-i)\mathrm{Q}^2(\lambda)\right] \tag{A.4}$$

Up to an overall prefactor $\mathrm{e}^{-2(M+m^1+m^2)\gamma\lambda}$, both left and right hand sides are polynomials in $\mathrm{e}^{2\gamma\lambda}$ of degree at most $M+m^1+m^2$. Furthermore they have the following $2M$ known zeroes: $\lambda = \mu_k \pm i/2$, $k = 1, \ldots, M$. If some μ_k coincide the zeroes have a multiplicity; however note that a $\mu_k + i/2$ cannot coincide with a $\mu_l - i/2$ (since the μ_k are not allowed to form strings). There are now two situations:
1) $m^1 + m^2 < M$. In this case we conclude directly that both sides of Eq. (A.4) are zero.
2) $m^1 + m^2 = M$. Since $m^1 \leq M/2$ and $m^2 \leq M/2$, this can only happen if $m^1 = m^2 = M/2$. However in this case direct computation of the highest degree terms of the polynomials in Eq. (A.4) shows that they are zero, and therefore they are in fact of degree at most $2M-1$. Again this means that both sides of the equation are zero.

In either case, we finally find

$$\frac{\mathrm{Q}^1(\lambda+i)}{\mathrm{Q}^1(\lambda)} = \frac{\mathrm{Q}^2(\lambda+i)}{\mathrm{Q}^2(\lambda)} \qquad \forall\lambda \tag{A.5}$$

If q is not a root of unity, this implies immediately that $\{\lambda_j^1\} = \{\lambda_j^2\}$. What we have proven is the following result:

A.1. Two Bethe states with $m \leq M/2$ (i.e. $S^z \geq 0$) correspond to the same eigenvalues of the $\mathbf{T}(\lambda)$ (for all λ) if and only if they are identical.

This has the following two immediate corollaries:

A.2. Two distinct Bethe states are orthogonal to each other.

A.3. A Bethe state with $m = M/2$ (i.e. $S^z = 0$) is an eigenstate of the flip operator $\mathbf{R}$.

Remark: the situation is much more subtle if q is a root of unity. One way to see this is to consider Eq. (A.5), and assume now that $q^{2N} = 1$. One can add an extra "full string" of the form $\lambda_i = \lambda_0 + ij$, $j = 1, \ldots, N$ to one of the sets without modifying the corresponding Q function. This suggests extra degeneracy can appear when q is a root of unity between states with $\Delta S^z = N$, which is precisely the phenomenon observed in [21].

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